1.2 Vector bundles and connections

Definition 1.2.1. Let X and Y be complex manifolds. A continuous map $f : X \to Y$ is a **holomorphic** map if for any holomorphic charts (U, φ) and (U', φ') of X and Y, respectively, the map $\varphi' \circ f \circ \varphi : \varphi(f^{-1}(U') \cap U) \to \varphi'(U')$ is holomorphic.

Definition 1.2.2. Let M be a complex manifold and E be a complex vector bundle over M. We say that E is a **holomorphic** vector bundle if for any i, j such that $U_i \cap U_j \neq \emptyset$, ψ_{ij} in (1.1.3) is a holomorphic map.

Remark that the complex vector bundle could be defined over any manifolds, but the holomorphic vector bundle is only well-defined over complex manifolds.

It is easy to see that the total space of a holomorphic vector bundle is a complex manifold.

Proposition 1.2.3. *The complex vector bundle $T^{(1,0)}M$ over M is holomorphic.*

Proof. The proposition follows the fact that the transition map for $T^{(1,0)}M$ is the same as that of complex manifold M. □

Since $T^{(1,0)}M$ is locally spanned by $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$, we will also regard it as the complex tangent bundle of M.

Example 1.2.4. Any canonical construction in linear algebra gives rise to a geometric version for complex (resp. holomorphic) vector bundles. Let E and F be complex (resp. holomorphic) vector bundles over M.

- The direct sum $E \oplus F$ is the complex (resp. holomorphic) vector bundle over M such that the fibre $(E \oplus F)|_x$ for any $x \in M$ is canonically isomorphic to $E|_x \oplus F|_x$ as complex vector spaces.
- The tensor product $E \otimes F$ is the complex (resp. holomorphic) vector bundle over M such that the fibre $(E \otimes F)|_x$ for any $x \in M$ is canonically isomorphic to $E|_x \otimes F|_x$ as complex vector spaces.
- The i-th exterior power $\Lambda^i E$ and the i-th symmetric power $S^i E$ are the complex (resp. holomorphic) vector bundle over M such that the fibres for any $x \in M$ are canonically isomorphic to $\Lambda^i(E|_x)$ and $S^i(E|_x)$ respectively.
- The dual bundle E^* is the complex (resp. holomorphic) vector bundle over M such that the fibre $E^*|_x$ for any $x \in M$ is canonically isomorphic to $(E|_x)^*$.

- The endomorphism bundle $\mathrm{End}(E)$ is the complex (resp. holomorphic) vector bundle over M such that the fibre $\mathrm{End}(E)|_x$ for any $x \in M$ is canonically isomorphic to $\mathrm{End}(E|_x)$.

Proposition 1.2.5. *The set $\gamma_n \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ that consists of all pairs $(\ell, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$ with $z \in \ell$ forms in a natural way a holomorphic line bundle over $\mathbb{CP}^n$. It is called the* ***tautological line bundle*** *over $\mathbb{CP}^n$.*

Proof. The projection $\pi : \gamma_n \to \mathbb{CP}^n$ is given by projecting to the first factor. Let $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ be the standard open covering in (1.1.26). Let $\ell = [z_0 : \cdots : z_n]$. A canonical trivialization of γ_n over U_i is given by

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \quad (\ell, z) \mapsto (\ell, z_i). \tag{1.2.1}$$

Then the transition maps $\ell \mapsto z_i/z_j$ is holomorphic. □

Let E be a complex vector bundle over a smooth manifold M. A linear map

$$\nabla^E : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, T^*M \otimes E) \tag{1.2.2}$$

is called a **connection** on E if for any $\varphi \in \mathscr{C}^\infty(M, \mathbb{C})$, $s \in \mathscr{C}^\infty(M, E)$ and vector field V, we have

$$\nabla^E_V(\varphi s) = V(\varphi)s + \varphi \nabla^E_V s. \tag{1.2.3}$$

Connections on E always exist. Indeed, let $\{U_k\}_{k\in I}$ be an open covering of M such that $E|_{U_k}$ is trivial for any $k \in I$. If $\{\xi_{kl}\}_{l=1,\cdots,r}$ is a local frame of $E|_{U_k}$, any section $s \in \mathscr{C}^\infty(U_k, E)$ has the form $s = \sum_{l=1}^r s_l \xi_{kl}$ with uniquely determined $s_l \in \mathscr{C}^\infty(U_k)$. We define a connection on $E|_{U_k}$ by $\nabla^E_k s := \sum_{l=1}^r ds_l \otimes \xi_{kl}$. Consider now a partition of unity $\{\psi_k\}_{k\in I}$ subordinated to $\{U_k\}_{k\in I}$. Then $\nabla^E s := \sum_k \nabla^E_k(\psi_k s)$, $s \in \mathscr{C}^\infty(M, E)$, defines a connection on E.

If $'\nabla^E$ is another connection on E, then by (1.2.3), $'\nabla^E - \nabla^E \in \Omega^1(M, \mathrm{End}(E))$.

If ∇^E is a connection on E, then there exists a unique extension $\nabla^E : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ verifying the Leibniz rule: for any $\alpha \in \Omega^k(M, \mathbb{C})$, $s \in \Omega^*(M, E)$, then

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge \nabla^E s. \tag{1.2.4}$$

Proposition 1.2.6. *Let $(\nabla^E)^2 := \nabla^E \circ \nabla^E : \mathscr{C}^\infty(M, E) \to \Omega^2(M, E)$. For $s \in \mathscr{C}^\infty(M, E)$ and vector fields U, V on M, we have*

$$\left(\nabla^E\right)^2(U, V)s = \nabla^E_U \nabla^E_V s - \nabla^E_V \nabla^E_U s - \nabla^E_{[U,V]} s. \tag{1.2.5}$$

Proof. Let $\{e_i\}$ be a locally orthonormal frame of M and $\{e^i\}$ be its dual with respect to the metric. Then from (1.2.4),

$$\left(\nabla^E\right)^2 s = \nabla^E\left(e^j \otimes \nabla^E_{e_j} s\right) = de^j \otimes \nabla^E_{e_j} s + e^i \wedge e^j \otimes \nabla^E_{e_i}\nabla^E_{e_j} s. \tag{1.2.6}$$

Since

$$de^j(U,V) = U(e^j(V)) - V(e^j(U)) - e^j([U,V])$$

and

$$e^i \wedge e^j(U,V) = g(U,e_i)g(V,e_j) - g(U,e_j)g(V,e_i), \tag{1.2.7}$$

we have

$$\begin{aligned}\left(\nabla^E\right)^2(U,V)s = U(g(V,e_j))\nabla^E_{e_j}s + g(V,e_j)\nabla^E_U\nabla^E_{e_j}s \\ - V(g(U,e_j))\nabla^E_{e_j}s - g(U,e_j)\nabla^E_V\nabla^E_{e_j}s - \nabla^E_{[U,V]}s \\ = \nabla^E_U\nabla^E_V s - \nabla^E_V\nabla^E_U s - \nabla^E_{[U,V]}s.\end{aligned} \tag{1.2.8}$$

The proof of this proposition is completed. □

Let R^E be the curvature of ∇^E. Then from Proposition 1.2.6, we have

$$(\nabla^E)^2 = R^E \in \Omega^2(M, \mathrm{End}(E)). \tag{1.2.9}$$

From the Leibniz's rule, the operator $(\nabla^E)^2$ and R^E could be extended to act on $\Omega^*(M,E)$. Moreover, they are also equal after the extension.

Proposition 1.2.7 (Bianchi Identity)**.** *The following identity holds,*

$$[\nabla^E, R^E] = 0. \tag{1.2.10}$$

Proof. Since $R^E = (\nabla^E)^2$,

$$[\nabla^E, R^E] = [\nabla^E, (\nabla^E)^2] = 0. \tag{1.2.11}$$

□

Let h^E be a **Hermitian metric** on E, i.e., a smooth family $\{h^E_x\}_{x\in M}$ of sesquilinear maps $h^E_x : E_x \times E_x \to \mathbb{C}$ such that $h^E_x(\xi,\xi) > 0$ for any $\xi \in E_x \backslash \{0\}$. We call (E, h^E) a Hermitian vector bundle on M. There always exist Hermitian metrics on E by using the partition of unity as above.

Example 1.2.8. By (1.1.21), for any $Z, Z' \in T^{(1,0)}M$,

$$h^{T^{(1,0)}M}(Z, Z') := g(Z, \overline{Z'}) \tag{1.2.12}$$

defines a Hermitian metric on $T^{(1,0)}M$. Let $h_{ij} = h^{T^{(1,0)}M}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right)$. Then by (1.1.20),

$$h_{ij} = g_{i\bar{j}}. \tag{1.2.13}$$

Definition 1.2.9. A connection ∇^E is said to be a **Hermitian connection** on (E, h^E) if for any $s_1, s_2 \in \mathscr{C}^\infty(M, E)$,

$$dh^E(s_1, s_2) = h^E(\nabla^E s_1, s_2) + h^E(s_1, \nabla^E s_2). \tag{1.2.14}$$

There always exist Hermitian connections. Indeed, let ∇_0^E be a connection on E, then $h^E(\nabla_1^E s_1, s_2) = dh^E(s_1, s_2) - h^E(s_1, \nabla_0^E s_2)$ defines a connection ∇_1^E on E. Then $\nabla^E = \frac{1}{2}(\nabla_0^E + \nabla_1^E)$ is a Hermitian connection on (E, h^E).

In the rest of this section, we assume that E is a holomorphic vector bundle over a complex manifold M.

Let

$$\Omega^{p,q}(M, E) := \mathscr{C}^\infty(M, \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M) \otimes E). \tag{1.2.15}$$

Any section $s \in \mathscr{C}^\infty(M, E)$ has the local form $s = \sum_i \varphi_i \xi_i$, where $\{\xi_i\}$ is a holomorphic frame of E and φ_l are smooth functions. We set

$$\overline{\partial}^E s = \sum_l (\overline{\partial}\varphi_i)\xi_i, \tag{1.2.16}$$

where $\overline{\partial}\varphi_i = \sum_j d\bar{z}^j \frac{\partial}{\partial \bar{z}_j}\varphi_i$ in holomorphic coordinates $(z_1, \cdots, z_n)$. Then the operator

$$\overline{\partial}^E : \mathscr{C}^\infty(M, E) \to \Omega^{0,1}(M, E) \tag{1.2.17}$$

in (1.2.16) is well-defined. Indeed, if $\{\xi'_j\}$ is another holomorphic basis and (ψ_{ij}) is the holomorphic transition matrix, i.e., $\xi_i = \sum_j \psi_{ij}\xi'_j$, then $s = \sum_j(\sum_i \varphi_i\psi_{ij})\xi'_j$ and in this coordinates,

$$\begin{aligned}\sum_j \overline{\partial}\left(\sum_i \varphi_i\psi_{ij}\right)\xi'_j &= \sum_j \overline{\partial}\left(\sum_i \varphi_i\psi_{ij}\right)\xi'_j \\ &= \sum_i \overline{\partial}\varphi_i \sum_j \psi_{ij}\xi'_j = \overline{\partial}^E s.\end{aligned} \tag{1.2.18}$$

Definition 1.2.10. A connection ∇^E on E is said to be a **holomorphic connection** if $\nabla^E_V s = i_V(\overline{\partial}^E s)$ for any $V \in T^{(0,1)}M$ and $s \in \mathscr{C}^\infty(M, E)$.

Let $\{\xi_l\}_{l=1,\cdots,r}$ be a local frame of E. Denote by $h = (h_{lk} = h^E(\xi_k, \xi_l))$ the matrix of h^E with respect to $\{\xi_l\}_{l=1,\cdots,r}$. Let $s_1 = \sum_k \varphi_{1k}\xi_k$, $s_2 = \sum_l \varphi_{2l}\xi_l$. Let $\varphi_i = (\varphi_{i1}, \cdots, \varphi_{ir})$ for $i = 0, 1$. Then

$$h^E(s_1, s_2) = \overline{\varphi_2} \cdot h \cdot \varphi_1^t = \langle h \cdot \varphi_1^t, \overline{\varphi_2}^t \rangle. \tag{1.2.19}$$

The connection form $\Gamma = (\Gamma^l_k)$ of ∇^E with respect to $\{\xi_l\}_{l=1,\cdots,r}$ is defined by, with local 1-forms Γ^l_k,

$$\nabla^E \xi_k = \Gamma^l_k \xi_l. \tag{1.2.20}$$

For $s = \sum_k \varphi_k \xi_k$, denote by $\Gamma = (\Gamma_{lk}(:= \Gamma^l_k))$:

$$\Gamma s = (\xi_1, \cdots, \xi_r) \cdot \Gamma \cdot \varphi_1^t. \tag{1.2.21}$$

Recall that $R^E = d\Gamma + \Gamma \wedge \Gamma$. If ∇^E is holomorphic, by Definition 1.2.10, $\Gamma(V) = 0$ for any $T^{(0,1)}M$.

Theorem 1.2.11. *There exists a unique holomorphic Hermitian connection ∇^E on (E, h^E), called the* ***Chern connection****. With respect to a local holomorphic frame, the connection matrix is given by*

$$\Gamma = h^{-1}\partial h. \tag{1.2.22}$$

Proof. From Definition 1.2.10, we only need to define ∇^E_U for $U \in T^{(1,0)}M$. Relation (1.2.14) implies for $V \in T^{(1,0)}M$, $s_1, s_2 \in \mathscr{C}^\infty(M, E)$,

$$V(h^E(s_1, s_2)) = h^E(\nabla^E_V s_1, s_2) + h^E(s_1, \nabla^E_{\overline{V}} s_2). \tag{1.2.23}$$

Since $\nabla^E_{\overline{V}} s = i_{\overline{V}}(\overline{\partial}^E s)$, the above equation defines ∇^E_V uniquely. Moreover, if $\{\xi_l\}_{l=1,\cdots,r}$ is a local holomorphic frame of E, by (1.2.19) and (1.2.21),

$$\langle \partial h \cdot \varphi_1^t, \overline{\varphi_2}^t \rangle = \langle h\Gamma \cdot \varphi_1^t, \overline{\varphi_2}^t \rangle. \tag{1.2.24}$$

Thus we get (1.2.22). □

Since E is holomorphic, similar to (1.2.4), by Leibniz's rule, the operator $\overline{\partial}^E$ extends naturally to $\overline{\partial}^E : \Omega^{*,*}(M, E) \to \Omega^{*,*+1}(M, E)$ and $(\overline{\partial}^E)^2 = 0$.

Let ∇^E be the Chern connection on (E, h^E). Then we have a decomposition

$$\nabla^E = (\nabla^E)^{1,0} + (\nabla^E)^{0,1} \tag{1.2.25}$$

such that

$$(\nabla^E)^{1,0} : \Omega^{*,*}(M,E) \to \Omega^{*+1,*}(M,E), \quad (\nabla^E)^{0,1} = \overline{\partial}^E. \tag{1.2.26}$$

From (1.2.23), $s_1, s_2 \in \mathscr{C}^\infty(M,E)$,

$$\begin{aligned} h^E\left(\left((\nabla^E)^{1,0}\right)^2 s_1, s_2\right) &= \partial h^E\left((\nabla^E)^{1,0}s_1, s_2\right) + h^E\left((\nabla^E)^{1,0}s_1, \overline{\partial}^E s_2\right) \\ &= \partial\left(\partial h^E(s_1,s_2) - h^E\left(s_1, \overline{\partial}^E s_2\right)\right) + \partial h^E\left(s_1, \overline{\partial}^E s_2\right) \\ &\quad - h^E\left(s_1, (\overline{\partial}^E)^2 s_2\right) = 0. \end{aligned} \tag{1.2.27}$$

So $\left((\nabla^E)^{1,0}\right)^2 = 0$ and

$$(\nabla^E)^2 = \overline{\partial}^E \circ (\nabla^E)^{1,0} + (\nabla^E)^{1,0} \circ \overline{\partial}^E. \tag{1.2.28}$$

Then the curvature

$$R^E \in \Omega^{1,1}(M, \operatorname{End}(E)). \tag{1.2.29}$$

If $\operatorname{rank}(E) = 1$, $\operatorname{End}(E)$ is trivial. Since R^E is skew-adjoint, it is canonically identified as a (1,1)-form on M, such that $\sqrt{-1}R^E$ is real.

Example 1.2.12 (Tautological line bundle on $\mathbb{CP}^n$)**.** Recall that in Proposition 1.2.5, the point on γ_n is $(\ell, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$. It is natural to define a Hermitian metric h on γ_n by $h(\ell, z) = |z|^2$. Now we study it in local coordinates. By (1.2.1), if h_i is the metric of h on U_i, then we have

$$h(\ell, z) = h_i(\ell) z_i \bar{z}_i = h_i(\ell)|z_i|^2. \tag{1.2.30}$$

So in the coordinates (U_i, θ), for $\ell = [z_0, \cdots, z_n]$,

$$h_i(\ell) = \frac{|z|^2}{|z_i|^2} = 1 + |\theta|^2. \tag{1.2.31}$$

By Theorem 1.2.11, the connection form of the Chern connection is

$$\Gamma = h_i^{-1} \frac{\partial h_i}{\partial \theta_k} d\theta^k = \frac{\bar{\theta}_k d\theta^k}{1+|\theta|^2}. \tag{1.2.32}$$

The curvature

$$R^{\gamma_n} = d\Gamma = -\frac{(1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l}{(1+|\theta|^2)^2} d\theta^k \wedge d\bar{\theta}^l. \tag{1.2.33}$$

By (1.1.32),

$$\omega_{FS} = -\sqrt{-1}R^{\gamma_n}. \tag{1.2.34}$$

Let ∇ be the Levi-Civita connection on (TM, g), which could be naturally extended complex linearly on $TX \otimes \mathbb{C}$.

Theorem 1.2.13. *Let M be a almost complex manifold with triple (g, J, ω). Then the following statements are equivalent.*

(1) (M, ω) is Kähler.

(2) the bundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are preserved by ∇.

(3) $\nabla J = 0$.

Proof. (2)$\Leftrightarrow$(3) is obvious.

(3) $\Longrightarrow$ (1): From (1.1.23),

$$\begin{aligned} N^J(U,V) &= \nabla_U V - \nabla_V U + J\nabla_{JU} V - J\nabla_V JU \\ &\qquad + J\nabla_U JV - J\nabla_{JV} U - \nabla_{JU} JV + \nabla_{JV} JU \\ &= J(\nabla_U J)V - J(\nabla_V J)U - (\nabla_{JU} J)V + (\nabla_{JV} J)U \end{aligned} \tag{1.2.35}$$

for vector fields U, V. So $\nabla J = 0$ implies $N^J = 0$. Since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, we have

$$\begin{aligned} (\nabla_U \omega)(V,W) &= U(\omega(V,W)) - \omega(\nabla_U V, W) - \omega(V, \nabla_U W) \\ &= U(g(JV,W)) - g(\nabla_U JV, W) - g(JV, \nabla_U W) = 0 \end{aligned} \tag{1.2.36}$$

for any vector fields U, V, W.

For any $\alpha \in \Omega^k(M)$ and vector fields $X_0, \cdots, X_k$, we could obtain that

$$d\alpha(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\alpha)(X_0, \cdots, \widehat{X_i}, \cdots, X_k). \tag{1.2.37}$$

From (1.2.36) and (1.2.37), we have $d\omega = 0$.

(1) $\Longrightarrow$ (3): Since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, for vector fields U, V, W, we have

$$\begin{aligned} d\omega(U,V,W) &= U(\omega(V,W)) + V(\omega(W,U)) + W(\omega(U,V)) \\ &\qquad - \omega([U,V],W) + \omega([U,W],V) - \omega([V,W],U) \\ &= U(g(JV,W)) + V(g(JW,U)) + W(g(JU,V)) - g(J(\nabla_U V - \nabla_V U), W) \\ &\qquad + g(J(\nabla_U W - \nabla_W U), V) - g(J(\nabla_V W - \nabla_W V), U) \\ &= g((\nabla_U J)V, W) - g((\nabla_V J)U, W) + g((\nabla_W J)U, V). \end{aligned} \tag{1.2.38}$$

Since $g(JU,V) + g(U,JV) = 0$ and $(\nabla_W J)J = -J(\nabla_W J)$, by (1.2.35) and (1.2.38), we have

$$\begin{aligned} d\omega(JU,V,W) + d\omega(U,JV,W) &= g((\nabla_{JU} J)V, W) - g((\nabla_V J)JU, W) \\ + g((\nabla_W J)JU, V) + g((\nabla_U J)JV, W) &- g((\nabla_{JV} J)U, W) + g((\nabla_W J)U, JV) \\ &= 2g((\nabla_W J)U, JV) - g(N^J(U,V), W). \end{aligned} \tag{1.2.39}$$

So $d\omega = 0$ and $N^J = 0$ imply $\nabla J = 0$.

Our theorem is completed. □

Proposition 1.2.14. *Let M be a complex manifold. Let $\nabla^{T^{(1,0)}M}$ be the Chern connection on $T^{(1,0)}M$. For any $v \in \mathscr{C}^\infty(M, T^{(0,1)}M)$, we define $\nabla^{T^{(0,1)}M} v := \overline{\nabla^{T^{(1,0)}M}\overline{v}}$. Set $\widetilde{\nabla} = \nabla^{T^{(1,0)}M} \oplus \nabla^{T^{(0,1)}M}$. Then (M, ω) is a Kähler manifold if and only if $\nabla = \widetilde{\nabla}$, which means that the restriction of the Levi-Civita connection on $T^{(1,0)}M$ is just the Chern connection.*

Proof. By defintion, the bundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are preserved by $\widetilde{\nabla}$. If $\nabla = \widetilde{\nabla}$, from Theorem 1.2.13, we know that (M, ω) is a Kähler.

If (M, ω) is a Kähler, by Theorem 1.2.13, the bundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are preserved by ∇. Since $\nabla v = \overline{\nabla \overline{v}}$, we only need to prove that the restriction of ∇ on $T^{(1,0)}M$ is holomorphic and Hermitian. Since ∇ is metric-preserving, we only need to prove that ∇ is holomorphic on $T^{(1,0)}M$.

Let V be a holomorphic vector field, $U \in \mathscr{C}^\infty(M, T^{(0,1)}M)$. Then $[U, V] \in \mathscr{C}^\infty(M, T^{(0,1)}M)$. In fact, for any holomorphic function f, $V(f)$ is holomorphic. Since $U(f) = 0$ and $U(V(f)) = 0$, we have $[U, V]f = 0$.

Since $U \in \mathscr{C}^\infty(M, T^{(0,1)}M)$, it has a decomposition $U = X + \sqrt{-1}JX$. So

$$J[X + \sqrt{-1}JX, V] = -\sqrt{-1}[X + \sqrt{-1}JX, V]. \tag{1.2.40}$$

It is equivalent to

$$(J + \sqrt{-1})([X, V] + J[JX, V]) = 0. \tag{1.2.41}$$

Thus $[X, V] + J[JX, V] \in T^{(0,1)}M$. On the other hand,

$$\begin{aligned}\nabla_U V = \nabla_X V + \sqrt{-1}\nabla_{JX} V = \nabla_X V + J\nabla_{JX} V\\ = [X, V] + J[JX, V] - J(\nabla_V J)X = [X, V] + J[JX, V].\end{aligned} \tag{1.2.42}$$

Since ∇ preserves $T^{(1,0)}M$, we have $[X, V] + J[JX, V] \in T^{(1,0)}M$. So

$$\nabla_U V = [X, V] + J[JX, V] = 0. \tag{1.2.43}$$

The proof of our proposition is completed. □

Theorem 1.2.15 (Normal coordinates)**.** *A complex manifold M with triple (g, J, ω) is Kähler if and only if around each point of M, there exist holomorphic coordinates in which $g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2)$.*

Proof. If $g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2)$, by (1.1.22),

$$d\omega = \sqrt{-1}\left(\frac{\partial g_{i\bar{j}}}{\partial x_k}dx^k + \frac{\partial g_{i\bar{j}}}{\partial y_k}dy^k\right) \wedge dz^i \wedge d\bar{z}^j = 0. \tag{1.2.44}$$

Conversely, let $(z^1, \cdots, z^n)$ be a holomorphic frame such that $g_{i\bar{j}}(0) = \delta_{ij}$. Then $g_{i\bar{j}} = \delta_{ij} + a_{ijk}z^k + a_{ij\bar{k}}\bar{z}^k + O(|z|^2)$. Since $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$, we have $a_{ij\bar{k}} = \overline{a_{jik}}$. Since

$$d\omega = \sqrt{-1}(a_{ijk}dz^k + a_{ij\bar{k}}d\bar{z}^k) \wedge dz^i \wedge d\bar{z}^j + O(|z|^2), \tag{1.2.45}$$

The Kähler condition implies that $a_{ijk}dz^k \wedge dz^i \wedge d\bar{z}^j = 0$. It means that $a_{ijk} = a_{kji}$.

We choose a local frame frame $(\theta^1, \cdots, \theta^n)$ by

$$z^i = \theta^i - \frac{1}{2}a_{kij}\theta^j\theta^k. \tag{1.2.46}$$

We claim that this frame is holomorphic. In fact, set $z = f(\theta)$. Then f is holomorphic. Observe that

$$0 = \frac{\partial(f^{-1} \circ f)}{\partial\bar{\theta}_j} = \frac{\partial f^{-1}}{\partial z_k}\frac{\partial f_k}{\partial\bar{\theta}_j} + \frac{\partial f^{-1}}{\partial\bar{z}_k}\frac{\partial\overline{f_k}}{\partial\bar{\theta}_j} = \frac{\partial f^{-1}}{\partial\bar{z}_k}\overline{\frac{\partial f_k}{\partial\theta_j}}. \tag{1.2.47}$$

Since $(\partial f_k/\partial\theta_j)$ is non-degenerate, we see that f^{-1} is holomorphic[1].

For this coordinate change,

$$dz^i = d\theta^i - a_{kij}\theta^j d\theta^k. \tag{1.2.48}$$

So in this new coordinates,

$$\begin{aligned}\omega &= \sqrt{-1}(\delta_{ij} + a_{ijk}z^k + a_{ij\bar{k}}\bar{z}^k + O(|z|^2))dz^i \wedge d\bar{z}^j \\ &= \sqrt{-1}(\delta_{ij} + a_{ijk}\theta^k + a_{ij\bar{k}}\bar{\theta}^k + O(|\theta|^2))(d\theta^i - a_{qil}\theta^l d\theta^q) \wedge (d\bar{\theta}^j - \overline{a_{pjs}}\bar{\theta}^s d\bar{\theta}^p) \\ &= \sqrt{-1}(\delta_{ij} + O(|\theta|^2))d\theta^i \wedge d\bar{\theta}^j. \end{aligned} \tag{1.2.49}$$

From (1.1.22), we have $g_{i\bar{j}}(\theta) = \delta_{ij} + O(|\theta|^2)$ □

In general, the normal coordinates in Riemannian geometry is different from that in Kähler geometry.

[1] This argument and the inverse function theorem for real function gives a proof of the following **inverse function theorem for holomorphic function**: Let $f : U \to V$ be a holomorphic map between two open subsets $U, V \subset \mathbb{C}^n$. If $z \in U$ is regular, i.e., the complex Jacobian $J(f)(z) = (\partial f_i/\partial z_j)$ is surjective, then there exist open subsets $z \in U' \subset U$ and $f(z) \in V' \subset V$ such that f induces a biholomorphic map $f : U' \to V'$.